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Cooperative dynamics of snowdrift game on spatial distance-dependent small-world networks

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Abstract. We investigate the evolution of cooperative behaviors of small-world networking agents in a snowdrift game mode, where two agents (nodes) are connected with probability depending on their spatial Euclidean lattice distance in the power-law form controlled by an exponent α. Extensive numerical simulations indicate that the game dynamics crucially depends on the spatial topological structure of underlying networks with different values of the exponent α. Especially, in the distance-independent case of $\alpha = 0$, the small-world connectivity pattern contributes to an enhancement of cooperation compared with that in regular lattices, even for the case of having a high cost-to-benefit ratio r. However, with the increment of $\alpha > 0$, when $r \geq 0.4$, the spatial distance-dependent small-world (SDSW) structure tends to inhibit the evolution of cooperation in the snowdrift game.

PACS. 02.50.Le Decision theory and game theory – 87.23.Kg Dynamics of evolution – 89.75.Fb Structures and organization in complex systems

1 Introduction

Understanding the game mechanisms responsible for the emergence and persistence of cooperative behaviors has become one of the central problems in evolutionary biology and socioeconomics [1,2]. Game theory [3] and the theory of evolutionary games [4,5] provide a sufficient framework to model individual interactions. Especially, the snowdrift game (SG) [6,7], an alternative to the prisoner's dilemma game [8] for studying cooperation, has attracted considerable interests.

The SG, also known as the Hawk-Dove game, is originally a two-agent symmetric game, in which each agent can decide to take one of two strategies: cooperate (C) or defect (D). There are four possible combinations: (C, C), (C, D), (D, C), and (D, D) with their payoffs (R,R), (S,T), (T,S), and (P, P), respectively, satisfying the ordering condition $T > R > S > P$. The best action of one agent depends on his opponent: C is a better strategy than D if his opponent plays D; on the other hand, if his opponent plays C, then D is the best response. Considering now not just two agents but rather a large mixing population of agents, the game finally leads to a mixed evolutionarily stable state [9].

Several years later than the pioneering work of Nowak and May [10], it has been argued that the embedment of spatial interactive structure significantly affects the cooperative behaviors of the games. Recent studies of the SG played on two-dimensional lattices have shown that, the incorporation of spatial dimensional structure may inhibit cooperation [11], and the equilibrium proportion of cooperators is less than that expected by the game with replicator dynamics [12]. In a more recent work of the SG on a two-dimensional lattice, however, Sysi-Aho et al. proposed a different conclusion that, using a simple local decision rule, cooperation persists through the whole temptation parameter range [13]. The same results have also been obtained by Santos et al. in the case of the SG on scale-free networks, where they concluded that increasing heterogeneity of the network favors the emergence of cooperation [14,15]. The viewpoints that cooperation is sometimes inhibited and sometimes enhanced are also observed on several categories of complex networks ranging from regular lattices, small-world networks to random graphs, where the differences are due to different update rules and network contacts [16].

In many biological and social systems, individuals are often embedded in a Euclidean geographical space, and the interactions among them usually depend on their spatial distances. A typical example is neural networks in which spatial distances have great influences on the communications between neural cells [17]. In recent years, spatial Euclidean small-world networks have

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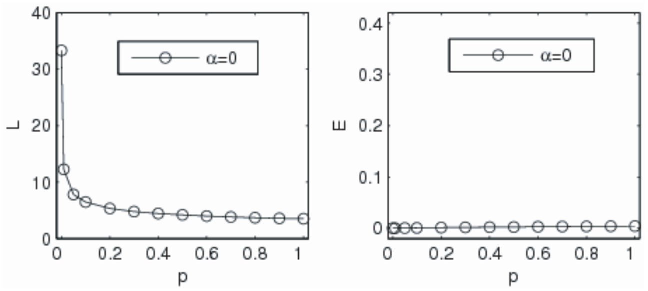

Fig. 1. The average path length L and the clustering coefficient E as a function of the probability p for small-world networks with $\alpha = 0$. Each point represents an average value over 100 runs simulated on 50×50 small-world networks.

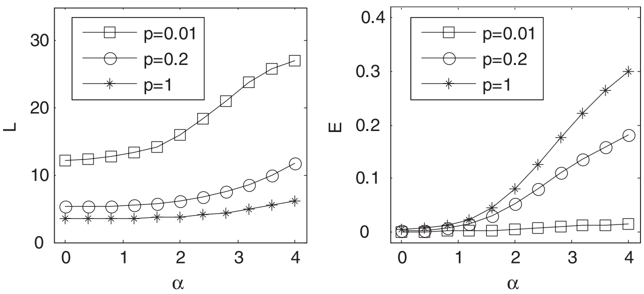

Fig. 2. The average path length L and the clustering coefficient E as a function of the exponent α for small-world networks with $p = 0.01, 0.2, 1$. Each point represents an average value over 100 runs simulated on 50×50 small-world networks.

attracted considerable interests, including but not limited to the navigability [18] and the nature of random walks [19–21]. Moving to the SG, we are interested in uncovering the dependence of cooperative behaviors on the spatial Euclidean structure embedded in underlying small-world networks. In this paper, a spatial distance-dependent small-world (SDSW) network topology is introduced to the SG, in which the length distribution of shortcuts is not uniform, and two nodes are connected according to their spatial lattice distance [22]. We try to explore the cooperative dynamics of the SG whose agents are in the spatial Euclidean situation.

2 SDSW network topology

The underlying SDSW network is constructed in a simplified Euclidean space [22]. To each site of a regular two-dimensional lattice, we assume it links to l nearest neighbors (the local contacts), where the constant $l \geq 0$. For a universal constant $q \geq 0$, we add edges with probability p from an arbitrary node u to other q nodes (the long-range contacts). Inspired by the Kleinberg small-world model [18], we select node v as one of the endpoints of q long-range connections of node u with the probability proportional to $[d(u,v)]^{-\alpha}$, where d is the lattice distance between the two nodes and the exponent $\alpha \geq 0$. Therefore, when $\alpha = 0$, one node's long-range connections are chosen with uniform probability independently of their positions on the lattice. As α increases, the long-range connections will be clustered in its vicinity and two distant nodes are less likely to be connected due to the distance-dependent cost of the edges. Such a graph has almost pqn^2 shortcuts and an average degree $\langle k \rangle \approx l + 2pq$ [23].

The exponent α has a strong impact on both local and global properties of the resulting network. For simplicity, we fix $l = q = 4$, i.e., each node has four nearest neighbors and $4p$ long-range neighbors. We observe the average path length L and the clustering coefficient E as functions of the probability p for 50×50 small-world networks with $\alpha = 0$. We can see from Figure 1 that at first L drops rapidly with the increase of p up to $p \approx 0.2$, at which point $\langle k \rangle \approx 6$, then keeps decreasing slowly with increased p. However, E keeps close to zero as p varies, implying that the network with $\alpha = 0$ has almost no local clustering property. For comparison, we plot L and E as functions of α in Figure 2 for 50×50 small-world networks with $p = 0.01, 0.2, 1$, respectively, where both L and E grow linearly with the increment of α, indicating the sensitivity of E with respect to α, i.e., the network becomes more and more densely clustered, at the same time the average path length of the network also becomes larger.

3 Spatial snowdrift game model

Following reference [11], we rescale the game such that it depends on a single parameter. In the model, we make $T = b > 1$, $R = b - 1/2$, $S = b - 1$ and $P = 0$, such that the cost-to-benefit ratio of mutual cooperation can be written as $r = 1/(2b - 1)$, where $0 \leq r \leq 1$.

We define that a round of play consists of the encounters of all pairs of individual x and its connected neighbor y, the payoff obtained from game interactions being stored as P_x. The payoffs earned by the agents are not accumulated from round to round. Whenever a site x is updated, a neighbor y is drawn at random among all k_x neighbors; whenever $P_y > P_x$, the chosen neighbor takes over site x with probability given by $(P_y - P_x)/(k_>(T - P))$, where $k_> = \max\{k_x, k_y\}$. In the process of evolution, updating is synchronous where all sites are updated simultaneously through competition with a randomly chosen neighbor.

4 Simulation results

After extensive numerical simulations, the instance of 50×50 small-world networks is selected as the illustration in the following[1]. We still take $l = q = 4$. The spatial networks are initialized randomly so that each node contains a cooperator or defector with the equal probability. Equilibrium frequencies of cooperation are obtained by averaging over 1000 generations after a transient time of 10 000 generations. All data are averaged over 100 groups of network realizations. The network connections, once

[1] We have done extensive numerical simulations under different system sizes, and found that all the qualitative conclusions of this paper still hold, which are independent of the system size.

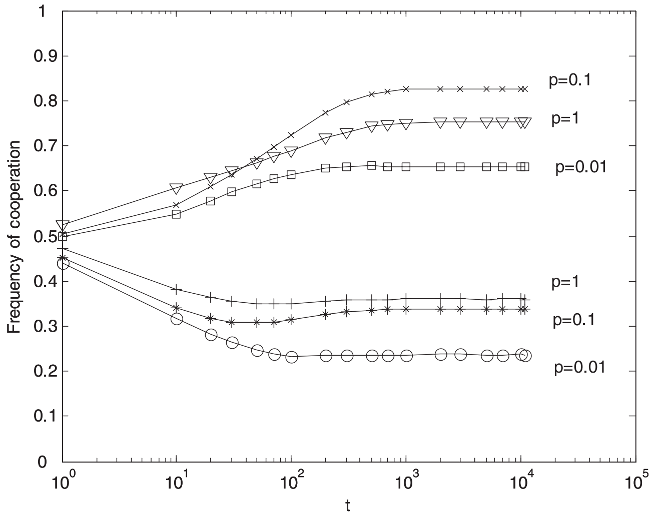

Fig. 3. The time evolution of the frequency of cooperation with the exponent $\alpha = 0$. The upper curves are for $r = 0.3$ and the lower curves for $r = 0.6$. In both cases the probability is varied as $p = 0.01, 0.1, 1$. Each point represents an average value over 100 runs simulated on 50×50 small-world networks.

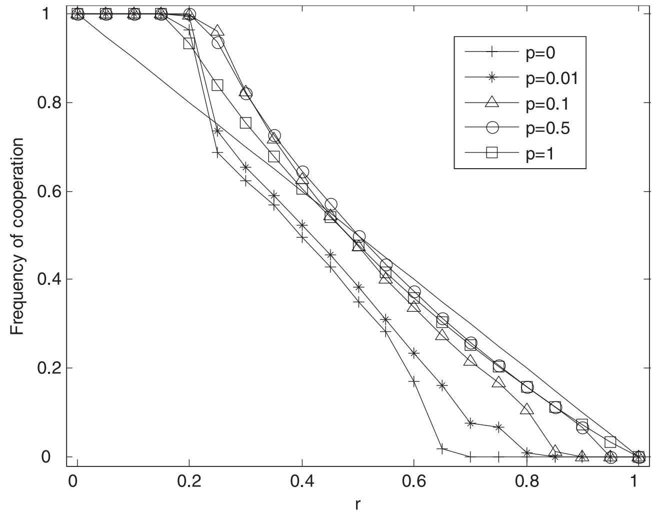

Fig. 4. Frequency of cooperation as a function of the cost-to-benefit ratio r in the snowdrift game on small-world networks with the exponent $\alpha = 0$ and the probability $p = 0, 0.01, 0.1, 0.5, 1$. Each point represents an average value over 100 runs simulated on 50×50 small-world networks.

generated, remain static throughout the evolution of the game.

First, we explore the effect of distance-independent small-world network mechanism with the exponent $\alpha = 0$ on the generic behaviors of the game. Figure 3 shows the time evolution of the frequency of cooperation F_c in spatial populations for $p = 0.01, 0.1, 1$, and $r = 0.3, 0.6$, respectively. In all the studied cases, the cooperation frequency F_c turns out to converge quite rapidly to a constant value without oscillations. We can see that, with the same r, the stable frequencies with different p are obviously distinguishable from each other, i.e., $F_c \sim 0.65, 0.82, 0.75$ for $r = 0.3$, $p = 0.01, 0.1, 1$ and $F_c \sim 0.23, 0.33, 0.35$ for $r = 0.6$, $p = 0.01, 0.1, 1$, respectively, implying that network topology significantly affects the behaviors of cooperation at steady states.

Figure 4 shows the frequency of cooperation as a function of r in the SG on small-world networks with different $p = 0, 0.01, 0.1, 0.5, 1$. We can see that the frequency with the small-world topology ($p > 0$) is improved significantly compared with that of the regular lattice ($p = 0$). Clearly, the major contribution to such an improvement arises from the adding of shortcuts in the population structure (taking place with increasing p). Similar with that in reference [11], the frequency of cooperation is higher than the mean field result (the dotted line $1-r$) for small r, spatial structure favoring defectors for larger r. The threshold above which the proportion of defectors is higher than the mean filed result depends on the value of p. For large r, deviations from the mean field result are less pronounced with the higher values of p.

We plot the frequency of cooperation as a function of p for different r in Figure 5, since the small-world probability is important for the evolution of cooperation. Remarkably, we find that with the independence of r,

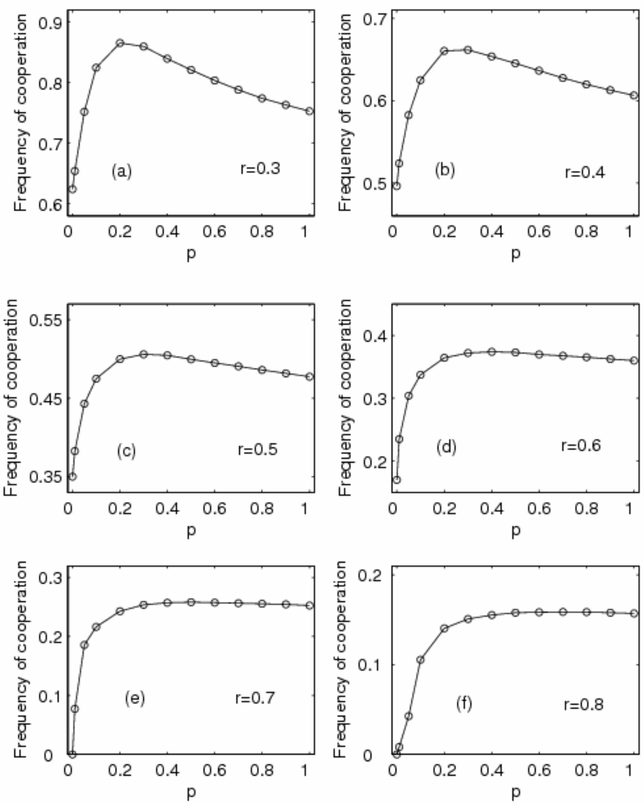

Fig. 5. (a–f) Frequency of cooperation as a function of the probability p in the snowdrift game on small-world networks with the exponent $\alpha = 0$ and the cost-to-benefit ratio $r = 0.3, 0.4, 0.5, 0.6, 0.7, 0.8$. Each point represents an average value over 100 runs simulated on 50×50 small-world networks.

the frequency keeps increasing with respect to the probability p up to $p \approx 0.2$, where the performance is already greatly enhanced. This indicates that the overall incidence of cooperators is sensitive to the average path length L, which has decreased typically one order of magnitude from its value at $p = 0$ as previously shown in Figure 1. For $r = 0.3$ and $r = 0.4$, there exists a pronounced frequency peak at $p \approx 0.2$, illustrating an optimal cooperation occurs at this point. With the increment of r, the peak becomes more and more saturated, and no further qualitative changes take place for $p > 0.2$, owing to the fact that L varies a little at this stage.

Now, we investigate how the exponent α in the SDSW networks influences the evolution of cooperation. As described above, the increment of α marks the onset of a more rapid change of the clustering property and a larger average path length. Only for small r ($r < 0.3$) is the proportion of cooperators not affected by the changes of α, mainly due to the high benefit and low cost. However, when r is larger, the spatial structure with increased α affects the cooperative dynamics greatly. The frequency of cooperation as a function of α is shown in Figure 6 for the SG with $p = 0.01, 0.2, 1$ and $r = 0.3, 0.6$, respectively. When $r < 0.4$, the benefit is lower but still relative high, which leads to complex behaviors with different p. A typical case is shown in the left panels of Figure 6 corresponding to $r = 0.3$, where as α grows from zero, the equilibrium proportion of cooperators decreases for $p = 1$, increases for $p = 0.2$, and remains almost unchanged for $p = 0.01$. When $r \geq 0.4$, the benefit becomes less, and the spatial clustered network makes cooperators be easily invaded by defectors, which favors defectors and tends to inhibit cooperation. A typical example is shown in the right panels of Figure 6 corresponding to $r = 0.6$, where the results are essentially identical for $p = 0.01, 0.2, 1$, respectively. We can observe that the equilibrium proportion of cooperators drops with the increment of α, where the long-distance connections become more and more sparse, and short-distance connections become more and more dense in the network, indicating long-distance connectivity pattern benefits overall cooperation in this high cost-to-benefit spatial situation. Our extensive simulations with other population sizes also support this turning point of cooperation at $r = 0.4$, which is independent of the population size.

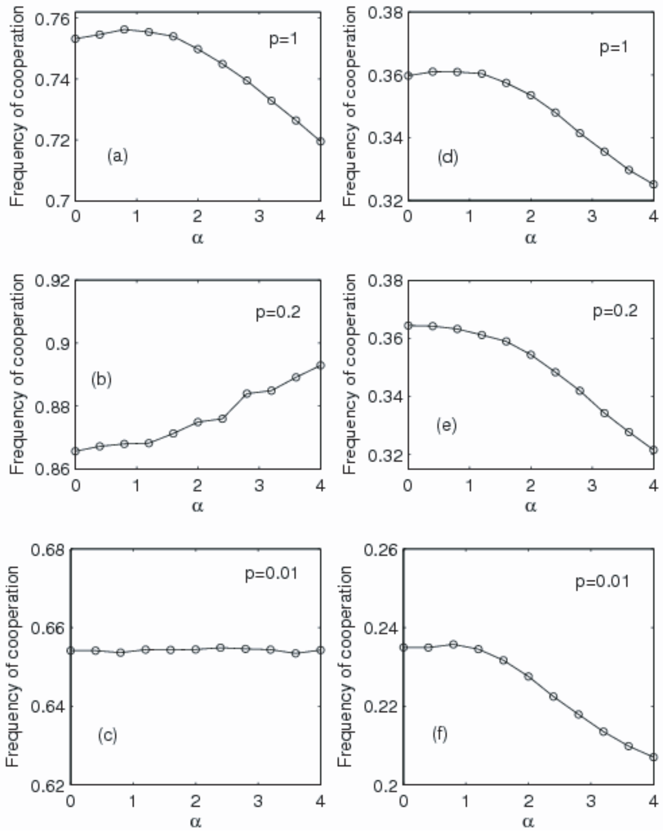

Fig. 6. Frequency of cooperation as a function of the exponent α for the snowdrift game on small-world networks with the probability $p = 0.01, 0.2, 1$ and the cost-to-benefit ratio $r = 0.3, 0.6$. (a–c) correspond to $r = 0.3$ and (d–f) Correspond to $r = 0.6$. Each point represents an average value over 100 runs simulated on 50×50 small-world networks.

5 Conclusion

In a summary, cooperative behaviors of the snowdrift game are sensitive to the underlying spatial network structures. In particular, when $\alpha = 0$, the spatial distance-independent small-world mechanism contributes to an enhanced level of cooperation among the populations compared with regular lattices, even with a very high cost-to-benefit ratio. When $\alpha > 0$, and $r \geq 0.4$, the SDSW network structure tends to inhibit the evolution of cooperation in the snowdrift game.

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